

# Stochastic Stability Properties of a Singularly Perturbed Chemical Langevin Equation

Narmada Herath and Domitilla Del Vecchio

Massachusetts Institute of Technology

March 18, 2014

## Introduction

In this report, we consider the model reduction of a set of singularly perturbed chemical Langevin equations, according to Theorem 3.1 in [1].

## Mathematical Notation and Terminology

We use  $\|\cdot\|$  to denote the euclidean norm and  $|\cdot|$  to denote the absolute value.

A continuous function  $\alpha : [0, a) \rightarrow [0, \infty)$ , is said to belong to class  $\mathcal{K}$  if it is strictly increasing and  $\alpha(0) = 0$ . It is said to belong to class  $\mathcal{K}_\infty$  if  $a = \infty$  and  $\alpha(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . [2]

A continuous function  $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ , is said to belong to class  $\mathcal{KL}$  if, for each fixed  $s$ , the mapping  $\beta(r, s)$  belongs to class  $\mathcal{K}$  with respect to  $r$  and, for each fixed  $r$ , the mapping  $\beta(r, s)$  is decreasing with respect to  $s$  and  $\beta(r, s) \rightarrow \infty$  as  $s \rightarrow \infty$ . [2]

## System Model

We consider a set of singularly perturbed chemical Langevin equations (1) - (4) with the singular perturbation parameter  $0 < \epsilon \ll 1$ ,  $x = (x_1, x_2)'$ ,  $z = (z_1, z_2)'$ , being the state variables :

$$\epsilon \dot{z}_1 = u - \delta_1 z_1 + \sqrt{\epsilon u} \Gamma_1 - \sqrt{\epsilon \delta_1 z_1} \Gamma_2, \quad (1)$$

$$\dot{x}_1 = \beta_1 z_1 - \delta_1(x_1 - z_2) + \sqrt{\beta_1 x_1} \Gamma_3 - \sqrt{\delta_1(x_1 - z_2)} \Gamma_4, \quad (2)$$

$$\epsilon \dot{z}_2 = b(x_1 - z_2) - a\delta_1 z_2 + \sqrt{\epsilon b(x_1 - z_2)} \Gamma_5 - \sqrt{a\epsilon \delta_1 z_2} \Gamma_6, \quad (3)$$

$$\dot{x}_2 = \beta_2 z_2 - \delta_2 x_2 + \sqrt{\beta_2 z_2} \Gamma_7 - \sqrt{\delta_2 x_2} \Gamma_8. \quad (4)$$

The equations (1) - (4), can be reduced to a system with  $\epsilon = 0$  according to Theorem 3.1 in [1]. The assumptions, given by (A1) - (A4), and the results of Theorem 3.1 in [1] are as follows.

Consider a set of singularly perturbed nonlinear Itô differential equations

$$\begin{aligned}\epsilon dz &= f_1(x, z, \theta, \epsilon)dt + g_1(x, z, \theta, \epsilon)dW_1, \\ dx &= f_2(x, z, \theta, \epsilon)dt + g_2(x, z, \theta, \epsilon)dW_2,\end{aligned}\tag{5}$$

where  $z \in \mathbb{R}^q, x \in \mathbb{R}^n$  and  $\theta \in \mathbb{R}^m$  is an input that is absolutely continuous, and  $\epsilon$  is a small positive constant (singular perturbation parameter).

**A1 :** The equation  $f_1(x, z, \theta, 0) = 0$  admits a unique solution  $z_s = h(x, \theta)$  which further satisfies  $g_1(x, z, \theta, 0) = 0$ . Moreover, the function  $h : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^p$  as well as its first and second derivatives are locally Lipschitz.

**A2 :** The reduced slow subsystem given by

$$dx = f_2(x, h(x, \theta), \theta, 0)dt + g_2(x, h(x, \theta), \theta, 0)dW_2,$$

is SISS with respect to input  $\theta \in \mathbb{R}^m$ , i.e.,  $\forall \nu > 0$ , there exist a class  $\mathcal{KL}$  function  $\beta(\cdot, \cdot)$  and class  $\mathcal{K}$  functions  $\gamma_1(\cdot)$ , such that

$$P \left\{ \|x(t)\| < \beta(\|x_0\|, t) + \gamma_1 \left( \sup_{0 \leq s \leq t} \|\theta(s)\| \right) \right\} \geq 1 - \nu, \quad \forall t \geq 0, \forall x_0 \in \mathbb{R}^2 \setminus \{0\}.$$

**A3 :** The reduced fast subsystem defined by

$$dy = f_1(x, h(x, \theta) + y, \theta, 0)d\tau + \tilde{g}_1(x, h(x, \theta) + y, \theta, 0)d\tilde{W}_1,$$

where  $\tilde{W}$  is a standard Wiener process on the fast time scale,  $\tilde{g} = \lim_{\epsilon \rightarrow 0} g_1/\sqrt{\epsilon}$  is assumed to be locally Lipschitz, and  $x, \theta$  are to be viewed as constants (on the fast time scale), is SISS with respect to state  $x \in \mathbb{R}^n$  and input  $\theta \in \mathbb{R}^m$ , i.e.,  $\forall \nu > 0$ , there exist a class  $\mathcal{KL}$  function  $\beta_y(\cdot, \cdot)$  and class  $\mathcal{K}$  functions  $\gamma_2^x(\cdot), \gamma_2(\cdot)$ , such that

$$P \left\{ \|y(t)\| < \beta(\|y_0\|, t) + \gamma_2^x \left( \sup_{0 \leq s \leq t} \|x(s)\| \right) + \gamma_2 \left( \sup_{0 \leq s \leq t} \|\theta(s)\| \right) \right\} \geq 1 - \nu, \quad \forall t \geq 0, \\ \forall z_0 \in \mathbb{R}^2 \setminus \{0\}.$$

**A4 :** There exist class  $\mathcal{K}_\infty$  function  $p_1$ , a class  $\mathcal{K}$  function  $\gamma_1^{\tilde{y}}$ , a non-increasing function  $b : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow (0, 1]$  and positive constants  $\delta_x, \delta$  such that

$$2\gamma_1^{\tilde{y}} \circ (I + p_1) \circ \frac{\gamma_2^x}{b(\delta_x, \delta)}(s) \leq s.$$

Then, given  $\nu > 0$ , there exist class  $\mathcal{KL}$  functions  $\delta_1, \delta_2$ , class  $\mathcal{K}$  functions  $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_1^\epsilon, \tilde{\gamma}_2^\epsilon$ , and a positive real number satisfying  $\max\{\|x_0\|, \|y_0\|, \|\theta\|, \|\dot{\theta}\|\} \leq \delta$  and  $0 < \epsilon < \epsilon^*$ , the following relationship holds:

$$P \left\{ \|x(t)\| < \delta_1(\|(x_0, y_0)\|, t) + \tilde{\gamma}_1 \left( \sup_{0 \leq s \leq t} \|\theta(s)\| \right) + \tilde{\gamma}_1^\epsilon(\epsilon), \right. \\ \left. \text{and } \|y(t)\| < \delta_2(\|(x_0, y_0)\|, t/\epsilon) + \tilde{\gamma}_2^\epsilon(\epsilon) + \tilde{\gamma}_2 \left( \sup_{0 \leq s \leq t} \|\theta(s)\| \right) \right\} \geq 1 - v, \quad \forall t \geq 0.$$

Using these results, we can see that  $\|y(t)\|$  is bounded in probability and the bound decreases as  $\epsilon \rightarrow 0$ . Therefore, as  $\epsilon$  becomes smaller,  $h(x, \theta)$  becomes a better approximation of  $z$ .

To apply Theorem 3.1, we write the system (1) - (4) in the form of a set of Itô differential equations, using the relation  $dW/dt = \Gamma$  where  $dW$  is Wiener increment. Since  $\Gamma_i$  are independent identical Gaussian white noise processes, these dynamics are given by

$$\epsilon dz_1 = (u - \delta_1 z_1)dt + \sqrt{\epsilon u + \epsilon \delta_1 z_1} dW_1, \quad (6)$$

$$dx_1 = (\beta_1 z_1 - \delta_1(x_1 - z_2))dt + \sqrt{\beta_1 x_1 + \delta_1(x_1 - z_2)} dW_2, \quad (7)$$

$$\epsilon dz_2 = (b(x_1 - z_2) - a\delta_1 z_2)dt + \sqrt{\epsilon b(x_1 - z_2) + a\epsilon \delta_1 z_2} dW_3, \quad (8)$$

$$dx_2 = (\beta_2 z_2 - \delta_2 x_2)dt + \sqrt{\beta_2 z_2 + \delta_2 x_2} dW_4, \quad (9)$$

and correspond to system (5), with  $f_1(x, z, \theta, \epsilon) = (u - \delta_1 z_1, b(x_1 - z_2) - a\delta_1 z_2)'$ ,  $f_2(x, z, \theta, \epsilon) = (\beta_1 z_1 - \delta_1(x_1 - z_2), \beta_2 z_2 - \delta_2 x_2)'$ ,  $g_1(x, z, \theta, \epsilon) = (\sqrt{\epsilon u + \epsilon \delta_1 z_1}, \sqrt{\epsilon b(x_1 - z_2) + a\epsilon \delta_1 z_2})'$  and  $g_2(x, z, \theta, \epsilon) = (\sqrt{\beta_1 x_1 + \delta_1(x_1 - z_2)}, \sqrt{\beta_2 z_2 + \delta_2 x_2})'$ .

In the following sections, we demonstrate that each of the assumptions (A1) - (A4) are satisfied for the system (6) - (9).

## Verification of A1

When  $\epsilon = 0$ , the equation  $\begin{bmatrix} u - \delta_1 z_1 \\ b(x_1 - z_2) - a\delta_1 z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  admits the unique solution,

$$z_s = \begin{bmatrix} \frac{u}{\delta_1} \\ \frac{bx_1}{b+a\delta_1} \end{bmatrix} = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix},$$

and with  $\epsilon = 0$ , we have  $\begin{bmatrix} \sqrt{\epsilon u} \Gamma_1 - \sqrt{\epsilon \delta_1 z_1} \Gamma_2 \\ \sqrt{\epsilon b(x_1 - z_2)} \Gamma_5 - \sqrt{a\epsilon \delta_1 z_2} \Gamma_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

Therefore, A1 is satisfied.

## Verification of A2

We can obtain the reduced slow subsystem by substituting  $h_1(x)$  and  $h_2(x)$  in (7) and (9) as follows:

$$\begin{aligned} dx_1 &= \left( \frac{\beta_1 u}{\delta_1} - \delta_1 \left( x_1 - \frac{bx_1}{b+a\delta_1} \right) \right) dt + \sqrt{\frac{\beta_1 u}{\delta_1} + \delta_1 \left( x_1 - \frac{bx_1}{b+a\delta_1} \right)} dW_2, \\ dx_2 &= \left( \frac{\beta_2 bx_1}{b+a\delta_1} - \delta_2 x_2 \right) dt + \sqrt{\frac{\beta_2 bx_1}{b+a\delta_1} + \delta_2 x_2} dW_4. \end{aligned}$$

Simplifying further, these dynamics are given by

$$\begin{aligned} dx_1 &= \left( \frac{\beta_1 u}{\delta_1} - \left( \frac{a\delta_1^2}{b+\delta_1} \right) x_1 \right) dt + \sqrt{\frac{\beta_1 u}{\delta_1} + \left( \frac{a\delta_1^2}{b+\delta_1} \right) x_1} dW_2, \\ dx_2 &= \left( \frac{\beta_2 bx_1}{b+a\delta_1} - \delta_2 x_2 \right) dt + \sqrt{\frac{\beta_2 bx_1}{b+a\delta_1} + \delta_2 x_2} dW_4. \end{aligned}$$

Let

$$\begin{aligned} A &= \begin{bmatrix} 0 & -\left( \frac{a\delta_1^2}{b+\delta_1} \right) \\ \frac{\beta_2 b}{b+a\delta_1} & -\delta_2 \end{bmatrix} = \begin{bmatrix} 0 & -\gamma \\ k_2 & -\delta_2 \end{bmatrix}, \\ g(x) &= \begin{bmatrix} \frac{\beta_1 u}{\delta_1} + \sqrt{\frac{\beta_1 u}{\delta_1} + \left( \frac{a\delta_1^2}{b+\delta_1} \right) x_1} \\ \sqrt{\frac{\beta_2 bx_1}{b+a\delta_1} + \delta_2 x_2} \end{bmatrix} = \begin{bmatrix} k_1 + \sqrt{k_1 + \gamma x_1} \\ \sqrt{k_2 x_1 + \delta_2 x_2} \end{bmatrix}. \end{aligned}$$

Then we obtain

$$dx = Axdx + \begin{bmatrix} \frac{\beta_1 u}{\delta_1} + \sqrt{\frac{\beta_1 u}{\delta_1} + \left( \frac{a\delta_1^2}{b+\delta_1} \right) x_1} \\ \sqrt{\frac{\beta_2 bx_1}{b+a\delta_1} + \delta_2 x_2} \end{bmatrix} \begin{bmatrix} dW_2 \\ dW_4 \end{bmatrix}, \quad (10)$$

$$dx = Axdx + g(x)dW_x. \quad (11)$$

where  $dW_x = (dW_2, dW_4)'$ . To prove that the system described by equation (10) is SISS with respect to an input  $\theta \in \mathbb{R}^2$  that will be defined at the end of this section, we proceed by using a change of coordinates such that  $v = P^{-1}x$ , with  $A = PDP^{-1}$  where  $D$  is a diagonal matrix. Specifically, we have that

$$D = \begin{bmatrix} -\frac{\delta_2}{2} - \frac{\sqrt{\delta_2^2 - 4\gamma k_2}}{2} & 0 \\ 0 & -\frac{\delta_2}{2} + \frac{\sqrt{\delta_2^2 - 4\gamma k_2}}{2} \end{bmatrix} = \begin{bmatrix} -D_1 & 0 \\ 0 & -D_2 \end{bmatrix},$$

$$P = \begin{bmatrix} \frac{\delta_2}{k_2} - \frac{1}{k_2} \left[ \frac{\delta_2}{2} + \frac{\sqrt{\delta_2^2 - 4\gamma k_2}}{2} \right] & \frac{\delta_2}{k_2} - \frac{1}{k_2} \left[ \frac{\delta_2}{2} - \frac{\sqrt{\delta_2^2 - 4\gamma k_2}}{2} \right] \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 \\ 1 & 1 \end{bmatrix},$$

$$P^{-1} = \begin{bmatrix} \frac{-k_2}{\sqrt{\delta^2 - 4\gamma k_2}} & \frac{\delta_2^2 + \sqrt{\delta^2 - 4\gamma k_2}}{2\sqrt{\delta^2 - 4\gamma k_2}} \\ \frac{k_2}{\sqrt{\delta^2 - 4\gamma k_2}} & \frac{-\delta_2^2 + \sqrt{\delta^2 - 4\gamma k_2}}{2\sqrt{\delta^2 - 4\gamma k_2}} \end{bmatrix} = \begin{bmatrix} F_1 & F_3 \\ F_2 & F_4 \end{bmatrix}.$$

Then,

$$x = Pv = \begin{bmatrix} P_1 & P_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} P_1 v_1 + P_2 v_2 \\ v_1 + v_2 \end{bmatrix},$$

$$dv = Dzdt + P^{-1}g(x)dW_x,$$

$$dv = \left( \begin{bmatrix} -D_1 & 0 \\ 0 & -D_2 \end{bmatrix} v + \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} k_1 \\ 0 \end{bmatrix} \right) dt + \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} \sqrt{k_1 + \gamma x_2} \\ \sqrt{k_2 x_1 + \delta_2 x_2} \end{bmatrix} \begin{bmatrix} dW_2 \\ dW_4 \end{bmatrix}. \quad (12)$$

We use Proposition 2.3 in [1] to prove that the slow subsystem defined by equation (12) is SISS with respect to an appropriate input  $\theta$  to be defined later. Consider the Lyapunov function  $V = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$ . Then,

$$\begin{aligned} LV &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \left( Dv + \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} k_1 \\ 0 \end{bmatrix} \right) \\ &+ \frac{1}{2} Tr \left\{ \begin{bmatrix} \sqrt{k_1 + \gamma x_1} & \sqrt{k_2 x_1 + \delta_2 x_2} \end{bmatrix} \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} F_1 & F_3 \\ F_2 & F_4 \end{bmatrix} \begin{bmatrix} \sqrt{k_1 + \gamma x_1} \\ \sqrt{k_2 x_1 + \delta_2 x_2} \end{bmatrix} \right\}, \\ &= -D_1 v_1^2 - D_2 v_2^2 + F_1 k_1 v_1 + F_3 k_1 v_2 \\ &\quad + \frac{1}{2} ((F_1^2 + F_2^2)(k_1 + \gamma x_1) + 2(F_1 F_3 + F_2 F_4) \sqrt{k_2 x_1 + \delta_2 x_2} \sqrt{k_1 + \gamma x_1}) \\ &\quad + \frac{1}{2} ((F_3^2 + F_4^2)(k_2 x_1 + \delta_2 x_2)), \\ &= -D_1 v_1^2 - D_2 v_2^2 + F_1 k_1 v_1 + F_3 k_1 v_2 \\ &\quad + \frac{1}{2} ((F_1^2 + F_2^2)(k_1 + \gamma x_1)) + (F_1 F_3 + F_2 F_4) \sqrt{k_2 x_1 + \delta_2 x_2} \sqrt{k_1 + \gamma x_1} \\ &\quad + \frac{1}{2} ((F_3^2 + F_4^2)(k_2 x_1 + \delta_2 x_2)), \\ &= -D_1 v_1^2 - D_2 v_2^2 + F_1 k_1 v_1 + F_3 k_1 v_2 + \frac{1}{2} ((F_1^2 + F_2^2)(k_1 + \gamma(P_1 v_1 + P_2 v_2))) \\ &\quad + (F_1 F_3 + F_2 F_4) \sqrt{k_2(P_1 v_1 + P_2 v_2) + \delta_2(v_1 + v_2)} \sqrt{k_1 + \gamma(P_1 v_1 + P_2 v_2)} \\ &\quad + \frac{1}{2} ((F_3^2 + F_4^2)(k_2(P_1 v_1 + P_2 v_2) + \delta_2(v_1 + v_2))). \end{aligned}$$

Using that

$$\begin{aligned} &\sqrt{k_2(P_1 v_1 + P_2 v_2) + \delta_2(v_1 + v_2)} \sqrt{k_1 + \gamma(P_1 v_1 + P_2 v_2)} \\ &\leq \left( \frac{k_2(P_1 v_1 + P_2 v_2) + \delta_2(v_1 + v_2)}{2} + \frac{k_1 + \gamma(P_1 v_1 + P_2 v_2)}{2} \right), \end{aligned}$$

we can write

$$\begin{aligned}
LV &\leq -D_1 v_1^2 - D_2 v_2^2 + F_1 k_1 v_1 + F_3 k_1 v_2 + \frac{1}{2}((F_1^2 + F_2^2)(k_1 + \gamma(P_1 v_1 + P_2 v_2))) \\
&\quad + (F_1 F_3 + F_2 F_4) \left( \frac{k_2(P_1 v_1 + P_2 v_2) + \delta_2(v_1 + v_2)}{2} + \frac{k_1 + \gamma(P_1 v_1 + P_2 v_2)}{2} \right) \\
&\quad + \frac{1}{2}((F_3^2 + F_4^2)(k_2(P_1 v_1 + P_2 v_2) + \delta_2(v_1 + v_2))) \\
&\leq -D_1 v_1^2 - D_2 v_2^2 + \\
&\quad \left( F_1 k_1 + \frac{1}{2}((F_1^2 + F_2^2)\gamma P_1 + (F_1 F_3 + F_2 F_4)(k_2 P_1 + \delta_2 + \gamma P_1) + (F_3^2 + F_4^2)(k_2 P_1 + \delta_2)) \right) v_1 \\
&\quad + \left( F_3 k_1 + \frac{1}{2}((F_1^2 + F_2^2)\gamma P_2 + (F_1 F_3 + F_2 F_4)(k_2 P_2 + \delta_2 + \gamma P_2) + (F_3^2 + F_4^2)(k_2 P_2 + \delta_2)) \right) v_2 \\
&\quad + \frac{1}{2}((F_1^2 + F_2^2)k_1 + (F_1 F_3 + F_2 F_4)k_1).
\end{aligned}$$

Let

$$\begin{aligned}
E_1 &= \left( F_1 k_1 + \frac{1}{2}((F_1^2 + F_2^2)\gamma P_1 + (F_1 F_3 + F_2 F_4)(k_2 P_1 + \delta_2 + \gamma P_1) + (F_3^2 + F_4^2)(k_2 P_1 + \delta_2)) \right), \\
E_2 &= \left( F_3 k_1 + \frac{1}{2}((F_1^2 + F_2^2)\gamma P_2 + (F_1 F_3 + F_2 F_4)(k_2 P_2 + \delta_2 + \gamma P_2) + (F_3^2 + F_4^2)(k_2 P_2 + \delta_2)) \right), \\
E_3 &= \frac{1}{2}((F_1^2 + F_2^2)k_1 + (F_1 F_3 + F_2 F_4)k_1).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
LV &\leq -D_1 v_1^2 - D_2 v_2^2 + E_1 v_1 + E_2 v_2 + E_3, \\
LV &\leq -\frac{D_1}{2} v_1^2 - \frac{D_1}{2} \left( \left( v_1 - \frac{E_1}{D_1} \right)^2 - \left( \frac{E_1}{D_1} \right)^2 \right) - \frac{D_2}{2} v_2^2 - \frac{D_2}{2} \left( \left( v_2 - \frac{E_2}{D_2} \right)^2 - \left( \frac{E_2}{D_2} \right)^2 \right) + E_3, \\
LV &\leq -\frac{D_1}{2} v_1^2 - \frac{D_1}{2} \left( v_1 - \frac{E_1}{D_1} \right)^2 + \left( \frac{E_1^2}{2D_1} \right) - \frac{D_2}{2} v_2^2 - \frac{D_2}{2} \left( v_2 - \frac{E_2}{D_2} \right)^2 + \left( \frac{E_2^2}{2D_2} \right) + E_3.
\end{aligned}$$

We can show that  $LV \leq -\eta_1(v_1^2 + v_2^2)$ , with  $\eta_1 = (\min \frac{D_1}{2}, \frac{D_2}{2})/2$ , if

$$\left( \frac{D_1}{2} - \eta_1 \right) v_1^2 + \left( \frac{D_2}{2} - \eta_1 \right) v_2^2 \geq \left| \left( \frac{E_1^2}{2D_1} \right) + \left( \frac{E_2^2}{2D_2} \right) + E_3 \right|, \quad (13)$$

i.e.,

$$v_1^2 \geq \frac{\left| \left( \frac{E_1^2}{2D_1} \right) + \left( \frac{E_2^2}{2D_2} \right) \right|}{\left( \frac{D_1}{2} - \eta_1 \right)} \quad \text{and} \quad v_2^2 \geq \frac{|E_3|}{\left( \frac{D_2}{2} - \eta_1 \right)}.$$

Then,

$$\sqrt{v_1^2 + v_2^2} \geq \sqrt{\frac{\left| \left( \frac{E_1^2}{2D_1} \right) + \left( \frac{E_2^2}{2D_2} \right) \right|}{\left( \frac{D_1}{2} - \eta_1 \right)} + \frac{|E_3|}{\left( \frac{D_2}{2} - \eta_1 \right)}}. \quad (14)$$

Therefore, for an input  $\theta = \left( \frac{\left| \left( \frac{E_1^2}{2D_1} \right) + \left( \frac{E_2^2}{2D_2} \right) \right|}{\left( \frac{D_1}{2} - \eta_1 \right)} + \frac{|E_3|}{\left( \frac{D_2}{2} - \eta_1 \right)}, \frac{\left| \frac{\delta_1}{8} + u \right|}{\left( \frac{\delta_1}{2} - \eta_2 \right)} \right)'$ , we have that  $LV \leq -\eta_1(v_1^2 + v_2^2)$ , for  $\|v\| \geq \sqrt{\|\theta\|}$ , and therefore, according to Proposition 2.3 the system is SISS with input  $\theta$ .

Thus,  $\forall \nu > 0$ , there exist a class  $KL$  function  $\beta(\cdot, \cdot)$  and class  $K$  functions  $\gamma_1(\cdot)$ , such that

$$P \left\{ \|v(t)\| < \beta(\|v_0\|, t) + \gamma_1 \left( \sup_{0 \leq s \leq t} \|\theta(s)\| \right) \right\} \geq 1 - \nu, \quad \forall t \geq 0, \forall v_0 \in \mathbb{R}^2 \setminus \{0\}.$$

As  $\|v\|^2 = x^T (P^{-1})^T P^{-1} x$ , where  $(P^{-1})^T P^{-1}$  is a positive definite matrix, we have that

$$\lambda_{\min}((P^{-1})^T P^{-1}) \|x\|^2 \leq \|v\|^2 \leq \lambda_{\max}((P^{-1})^T P^{-1}) \|x\|^2,$$

where  $\lambda_{\max}$  and  $\lambda_{\min}$  are the maximum and minimum eigenvalues of  $((P^{-1})^T P^{-1})$ , respectively.

Therefore, according to the definition of SISS, we obtain that the slow subsystem in the original coordinates is SISS, i.e.,  $\forall \nu > 0$ , there exist a class  $KL$  function  $\beta_x(\cdot, \cdot)$  and class  $K$  function  $\gamma_{1x}(\cdot)$ , such that

$$P \left\{ \|x(t)\| < \beta_x(\|x_0\|, t) + \gamma_{1x} \left( \sup_{0 \leq s \leq t} \|\theta(s)\| \right) \right\} \geq 1 - \nu, \quad \forall t \geq 0, \forall x_0 \in \mathbb{R}^2 \setminus \{0\}.$$

where  $\beta_x(\cdot, \cdot) = \frac{\beta(\cdot, \cdot)}{\sqrt{\lambda_{\min}((P^{-1})^T P^{-1})}}$  and  $\gamma_{1x}(\cdot) = \frac{\gamma_1(\cdot)}{\sqrt{\lambda_{\min}((P^{-1})^T P^{-1})}}$ .

This satisfies A2.

## Verification of A3

To obtain the reduced fast system, we define  $y_1 = z_1 - h_1(x)$  and  $y_2 = z_2 - h_2(x)$ . Then the fast subsystem is given by

$$\begin{aligned} dy_1 &= (u - \delta_1(y_1 + h_1(x)))d\tau + \sqrt{u + \delta_1(y_1 + h_1(x))} dW_{y_1}, \\ dy_2 &= (bx_1 - b(y_2 + h_2(x)))d\tau - a\delta_1(y_2 + h_2(x)) + \sqrt{bx_1 - b(y_2 + h_2(x)) + a\delta_1(y_2 + h_2(x))} dW_{y_2}. \end{aligned}$$

Simplifying further, we obtain

$$dy_1 = -\delta_1 y_1 d\tau + \sqrt{2u + \delta_1 y_1} dW_{y_1}, \quad (15)$$

$$dy_2 = -(b + a\delta_1)y_2 d\tau + \sqrt{\left(\frac{2a\delta_1 bx_1}{b + a\delta_1}\right) - (b - a\delta_1)y_2} dW_{y_2}. \quad (16)$$

Proposition 2.3 in [1] can be used to prove that the reduced fast subsystem defined by the equation (15) - (16) is SISS with respect to the input  $\theta$  and slow variable  $x$ , by considering an input  $\theta_x = (\theta, x)'$ , with  $\theta$  defined in the previous section. Consider the Lyapunov function  $V(y) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2$ . Then,

$$\begin{aligned} LV &= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} -\delta_1 y_1 \\ -(b + a\delta_1)y_2 \end{bmatrix} \\ &\quad + \frac{1}{2} Tr \left\{ \begin{bmatrix} \sqrt{2u + \delta_1 y_1} & \sqrt{\left(\frac{2a\delta_1 bx_1}{b + a\delta_1}\right) - (b - a\delta_1)y_2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2u + \delta_1 y_1} \\ \sqrt{\left(\frac{2a\delta_1 bx_1}{b + a\delta_1}\right) - (b - a\delta_1)y_2} \end{bmatrix} \right\}, \\ LV &= -\delta_1 y_1^2 - (b + a\delta_1)y_2^2 + \frac{1}{2} \left[ 2u + \delta_1 y_1 + \left(\frac{2a\delta_1 bx_1}{b + a\delta_1}\right) - (b - a\delta_1)y_2 \right] \\ LV &= -\delta_1 y_1^2 - (b + a\delta_1)y_2^2 + u + \frac{\delta_1 y_1}{2} + \frac{a\delta_1 bx_1}{b + a\delta_1} - (b - a\delta_1)y_2 \\ LV &= -\frac{\delta_1}{2} y_1^2 - \frac{\delta_1}{2} (y_1^2 - y_1) - \frac{(b + a\delta_1)}{2} y_2^2 - \frac{(b + a\delta_1)}{2} \left( y_2^2 + \frac{(b - a\delta_1)}{(b + a\delta_1)} y_2 \right) + u + \frac{a\delta_1 bx_1}{b + a\delta_1}, \\ LV &= -\frac{\delta_1}{2} y_1^2 - \frac{\delta_1}{2} \left( \left( y_1 - \frac{1}{2} \right)^2 - \frac{1}{4} \right) - \frac{(b + a\delta_1)}{2} y_2^2 \\ &\quad - \frac{(b + a\delta_1)}{2} \left( \left( y_2 + \frac{(b - a\delta_1)}{2(b + a\delta_1)} \right)^2 - \left( \frac{(b - a\delta_1)}{2(b + a\delta_1)} \right)^2 \right) + u + \frac{a\delta_1 bx_1}{b + a\delta_1}, \\ LV &= -\frac{\delta_1}{2} y_1^2 - \frac{\delta_1}{2} \left( y_1 - \frac{1}{2} \right)^2 + \frac{\delta_1}{8} - \frac{(b + a\delta_1)}{2} y_2^2 \\ &\quad - \frac{(b + a\delta_1)}{2} \left( y_2 + \frac{(b - a\delta_1)}{2(b + a\delta_1)} \right)^2 - \left( \frac{(b - a\delta_1)}{2(b + a\delta_1)} \right)^2 + u + \frac{a\delta_1 bx_1}{b + a\delta_1}. \end{aligned}$$

We can show that  $LV \leq -\eta_2(y_1^2 + y_2^2)$  with  $\eta_2 = (\min(\frac{\delta_1}{2}, \frac{(b+a\delta_1)}{2}))/2$  if

$$\left( \frac{\delta_1}{2} - \eta_2 \right) y_1^2 + \left( \frac{(b + a\delta_1)}{2} - \eta_2 \right) y_2^2 \geq \left| \frac{\delta_1}{8} + u + \frac{a\delta_1 bx_1}{b + a\delta_1} \right|,$$

i.e.,

$$\left( \frac{\delta_1}{2} - \eta_2 \right) y_1^2 \geq \left| \frac{\delta_1}{8} + u \right| \quad \text{and} \quad \left( \frac{(b + a\delta_1)}{2} - \eta_2 \right) y_2^2 \geq \left| \frac{a\delta_1 bx_1}{b + a\delta_1} \right|.$$



Then,

$$y_1^2 + y_2^2 \geq \frac{\left| \frac{\delta_1}{8} + u \right|}{\left( \frac{\delta_1}{2} - \eta_2 \right)} + \frac{\left| \frac{a\delta_1 b x_1}{b+a\delta_1} \right|}{\left( \frac{(b+a\delta_1)}{2} - \eta_2 \right)}.$$

This condition will be satisfied if

$$\begin{aligned} \sqrt{y_1^2 + y_2^2} &\geq \max \left( \sqrt{\frac{2 \left| \frac{\delta_1}{8} + u \right|}{\left( \frac{\delta_1}{2} - \eta_2 \right)}}, \sqrt{\frac{2 \left| \frac{a\delta_1 b x_1}{b+a\delta_1} \right|}{\left( \frac{(b+a\delta_1)}{2} - \eta_2 \right)}} \right), \\ \sqrt{y_1^2 + y_2^2} &\geq \max \left( \sqrt{2} \sqrt{\frac{\left| \frac{\delta_1}{8} + u \right|}{\left( \frac{\delta_1}{2} - \eta_2 \right)}}, \sqrt{\frac{2 \left| \frac{a\delta_1 b}{b+a\delta_1} \right| |x_1|}{\left( \frac{(b+a\delta_1)}{2} - \eta_2 \right)}} \right). \end{aligned} \quad (17)$$

Given the input  $\theta = \left( \frac{\left| \left( \frac{E_1^2}{2D_1} \right) + \left( \frac{E_2^2}{2D_2} \right) \right|}{\left( \frac{D_1}{2} - \eta_1 \right)} + \frac{|E_3|}{\left( \frac{D_2}{2} - \eta_1 \right)}, \frac{\left| \frac{\delta_1}{8} + u \right|}{\left( \frac{\delta_1}{2} - \eta_2 \right)} \right)'$ , the state variable  $x = (x_1, x_2)'$  and  $\theta_x = (\theta, x)'$ , then a sufficient condition for satisfying (17) is given by

$$\sqrt{y_1^2 + y_2^2} \geq \max \left( \sqrt{2} \sqrt{\|\theta_x\|}, \sqrt{\frac{2 \left| \frac{a\delta_1 b}{b+a\delta_1} \right| \|\theta_x\|}{\left( \frac{(b+a\delta_1)}{2} - \eta_2 \right)}} \right)$$

which can be written as

$$\sqrt{y_1^2 + y_2^2} \geq \rho(\|\theta_x\|)$$

where  $\rho(\|s\|) = \max \left( \sqrt{2} \sqrt{\|s\|}, \sqrt{\frac{2 \left| \frac{a\delta_1 b}{b+a\delta_1} \right| \|s\|}{\left( \frac{(b+a\delta_1)}{2} - \eta_2 \right)}} \right).$

Then, applying Proposition 2.3, the reduced fast subsystem is SISS with respect to the input  $\theta$  and slow variable  $x$ , i.e,  $\forall \nu > 0$ , there exist a class  $\mathcal{KL}$  function  $\beta_y(\cdot, \cdot)$  and class  $\mathcal{K}$  functions  $\gamma_2^x(\cdot), \gamma_2(\cdot)$ , such that

$$P \left\{ \|y(t)\| < \beta(\|y_0\|, t) + \gamma_2^x \left( \sup_{0 \leq s \leq t} \|x(s)\| \right) + \gamma_2 \left( \sup_{0 \leq s \leq t} \|\theta(s)\| \right) \right\} \geq 1 - \nu, \quad \forall t \geq 0, \forall y_0 \in \mathbb{R}^2 \setminus \{0\}.$$

Therefore, A3 is satisfied.

## Verification of A4

We can also show that there exist a class  $K_\infty$  function  $p_1$ , a class  $K$  function  $\gamma_1^{\tilde{y}}$ , a non-increasing function  $b : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow (0, 1]$  and positive constants  $\delta_x, \delta$  such that

$$2\gamma_1^{\tilde{y}} \circ (I + p_1) \circ \frac{\gamma_2^x}{b(\delta_x, \delta)}(s) \leq s,$$

by defining the functions

$$\begin{aligned}\gamma_1^{\tilde{y}}(s) &= \frac{1}{2}s, \\ p_1(s) &= s, \\ \gamma_2^x(s) &= \frac{s}{4}, \\ b(\delta_x, \delta) &= 1.\end{aligned}$$

This satisfies A4.

## Conclusion

Applying Theorem 3.1 in [1], we can see that the bound on  $\|y(t)\|$  decreases as  $\epsilon \rightarrow 0$ . Therefore,  $z$  is better approximated by  $h(x, \theta)$  as  $\epsilon$  becomes smaller.

## References

- [1] C. Tang and T. Basar. “Stochastic stability of singularly perturbed nonlinear systems.” Decision and Control, 2001. Proceedings of the 40th IEEE Conference on. Vol. 1. IEEE, 2001.
- [2] H. K Khalil. Nonlinear systems. Vol. 3. Upper Saddle River: Prentice hall, 2002.